We answer the following question for 2×2 matrices.

Question. Why does the determinant have that formula?

Instead of defining the determinant by a formula (as we did in class), mathematicians prefer to define it via its properties. We will investigate these properties here (assuming we already know the formula), and then we will show that the properties we come up with completely determine the formula we saw in lecture.

Properties of the Determinant

We investigate the properties of the determinant assuming we already know that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Identity Matrix

First notice that the special matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, called the identity matrix, satisfies

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1.$$

Thus *the determinant of the identity matrix is* 1.

Swapping Columns

We next check how the determinant is affected by swapping columns of the matrix.

$$\det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = bc - ad = -(ad - bc) = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus *switching columns negates the determinant.*

Scaling Columns

We next check how the determinant behaves when one of the columns is multiplied by a constant.

$$\det \begin{bmatrix} sa & tb \\ sc & sd \end{bmatrix} = (sa)(td) - (tb)(sc) = st(ad) - st(bc) = st(ad - bc) = st \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus *multiplying a column by a constant multiplies the determinant by that constant.*

Adding Columns

We next check how the determinant behaves when one of the columns is obtained by adding two columns.

$$\det \begin{bmatrix} a_1 + a_2 & b \\ c_1 + c_2 & d \end{bmatrix} = (a_1 + a_2)d - b(c_1 + c_2) = (a_1 d - bc_1) + (a_2 d - bc_2) = \det \begin{bmatrix} a_1 & b \\ c_1 & d \end{bmatrix} + \det \begin{bmatrix} a_2 & b \\ c_2 & d \end{bmatrix}$$
$$\det \begin{bmatrix} a & b_1 + b_2 \\ c & d_1 + d_2 \end{bmatrix} = a(d_1 + d_2) - (b_1 + b_2)c = (ad_1 - b_1 c) + (ad_2 - b_2 c) = \det \begin{bmatrix} a & b_1 \\ c & d_1 \end{bmatrix} + \det \begin{bmatrix} a & b_2 \\ c & d_2 \end{bmatrix}$$

Thus *adding columns results in summing the determinants.*

Finding the Formula

I claim that the four properties we just discovered imply the formula we gave for the determinants. We will derive the formula for the 2×2 determinant from these four properties.

To make sure we don't get confused and accidentally use the determinant formula we learned in class, let's consider an unknown function named λ which takes a 2×2 matrix and produces a scalar from it. We will only assume that this unknown function λ has the four properties of the determinant we derived above. We first rewrite these properties below with some terminology for easy reference below.[1]

[1] These properties actually have other names, but these will do for us.

1. $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ (Normality Property)

2. $\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\lambda \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ (Swapping Property)

3. $\lambda \begin{bmatrix} sa & tb \\ sc & td \end{bmatrix} = st\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (Scaling Property)

4. $\lambda \begin{bmatrix} a_1 + a_2 & b \\ c_1 + c_2 & d \end{bmatrix} = \lambda \begin{bmatrix} a_1 & b \\ c_1 & d \end{bmatrix} + \lambda \begin{bmatrix} a_2 & b \\ c_2 & d \end{bmatrix}$ and $\lambda \begin{bmatrix} a & b_1 + b_2 \\ c & d_1 + d_2 \end{bmatrix} = \lambda \begin{bmatrix} a & b_1 \\ c & d_1 \end{bmatrix} + \lambda \begin{bmatrix} a & b_2 \\ c & d_2 \end{bmatrix}$ (Addition Property)

These properties are completely abstract, so the function we are working with doesn't have a formula that we know of yet. Let's try to find a formula for λ! Justifications are given in parentheses at the end of each line.

$$\begin{aligned}
\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \lambda \begin{bmatrix} a+0 & b \\ 0+c & d \end{bmatrix} && \text{(Arithmetic)} \\
&= \lambda \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \lambda \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} && \text{(Addition Property)} \\
&= \lambda \begin{bmatrix} a & b+0 \\ 0 & 0+d \end{bmatrix} + \lambda \begin{bmatrix} 0 & b+0 \\ c & 0+d \end{bmatrix} && \text{(Arithmetic)} \\
&= \left(\lambda \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) + \left(\lambda \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\right) && \text{(Addition Property)} \\
&= \lambda \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \lambda \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + \lambda \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} && \text{(Rearrange Terms)} \\
&= ad\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + cb\lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + ab\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + cd\lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} && \text{(Scaling Property)} \\
&= ad\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - cb\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ab\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + cd\lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} && \text{(Swapping Property)} \\
&= (ad - bc)\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ab\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + cd\lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} && \text{(Algebra)}
\end{aligned}$$

Notice that swapping the columns of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ results in the same matrix. In particular

$$\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = -\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = -\lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

by the Swapping Property. Both of these equations have the form $x = -x$, which has unique solution $x = 0$. Thus

$$\lambda \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0 \qquad \text{and} \qquad \lambda \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0,$$

and we complete our computation via the following steps below.

$$\begin{aligned}
\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (ad - bc)\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ab0 + cd0 && \text{(Work Above)} \\
&= (ad - bc)\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} && \text{(Arithmetic)} \\
&= (ad - bc)1 && \text{(Normality Property)} \\
&= ad - bc && \text{(Arithmetic)}
\end{aligned}$$

Thus we've showed that given the abstract function λ has the formula $\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. As λ was defined only in terms of those four properties we discovered, we see that *every function λ which satisfies our four properties has this formula*! In particular, we proved the following proposition (for 2×2 matrices).

Proposition. *The determinant is the unique matrix function satisfying the Normality, Swapping, Scaling, and Addition properties.*